

Solution of Plateau's problem

J. Harrison

Department of Mathematics
University of California, Berkeley

March 13, 2012

Abstract

Plateau's problem is to show the existence of an area minimizing surface with a given boundary, a problem posed by Lagrange in 1760. Experiments conducted by Plateau showed that an area minimizing surface can be obtained in the form of a film of oil stretched on a wire frame, and the problem came to be called Plateau's problem. Special cases have been solved by Douglas, Rado, Besicovitch, Federer and Fleming, and others. Federer and Fleming used the chain complex of integral currents with its continuous boundary operator, a Poincaré Lemma and good compactness properties to solve Plateau's problem for orientable, embedded surfaces. But integral currents cannot represent surfaces such as the Moebius strip or surfaces with triple junctions. In the class of varifolds, there are no existence theorems for a general Plateau problem. We use the chain complex of differential chains, a geometric Poincaré Lemma, and good compactness properties of the complex to solve Plateau's problem in such generality as to find the first solution which minimizes area taken from a collection of surfaces that includes all previous special cases, as well as all smoothly immersed surfaces of any genus type, orientable or nonorientable, and surfaces with multiple junctions. Our result holds for all dimensions and codimension one surfaces in \mathbb{R}^n .

The author was partially supported by the Miller Institute for Basic Research in Science and the Foundational Questions in Physics Institute.

Contents

1	Introduction	1
2	Differential chains of type B	4
2.1	Dirac chains	4
2.2	Mass norm	4
2.3	Difference chains and the B^r norm	5
2.4	Pushforward	6
2.4.1	Algebraic chains	7
2.5	Vector fields	8
3	Operators	8
3.1	Extrusion	8
3.2	Retraction	8
3.3	Boundary	9
3.4	Prederivative	9
3.4.1	The chainlet complex	11
4	Integral monopole and dipole chains	11
4.1	Volume functional for computing area of integral dipole chains	12
5	Geometric Poincaré Lemma	14
5.1	Cartesian wedge product	14
5.2	Geometric Poincaré Lemma	15
6	The part of a chain in an open set	16
7	Existence of area minimizers for surfaces spanning a Jordan curve in \mathbb{R}^3	20
7.1	A complete space of spanning dipole surfaces	20
7.1.1	Span	21
7.1.2	Existence of an integral dipole surface that spans γ	24

7.2	A lower bound on area	25
7.3	Compactness	25
	References	30

1 Introduction

Plateau’s problem asks whether there exists a surface with minimal area that spans a prescribed smooth Jordan curve γ . The solution depends on the definitions of “surface”, “area”, and “span”. Given a collection \mathcal{C} of surfaces, there is a natural sequence of questions:

1. Does there exist $S \in \mathcal{C}$ that spans γ ?
2. Is the infimum m of areas of surfaces spanning γ nonzero?
3. Does there exist a surface S_0 spanning γ with area m ?
4. What is the structure of S_0 away from γ ?
5. What is the structure of S_0 near γ ?

Historically, affirmative answers to (1) and (3) have been celebrated as a *solution to Plateau’s problem*, leaving questions of regularity (4) and (5) for further work over a period of time. Question (2) is often ignored, but Figure 6 shows that (2) is nontrivial and that one must pay attention to the definition of span, even when γ is a circle. The general problem of solving (1)-(3) for a collection \mathcal{C} containing all known soap¹ films arising in nature has been an open problem for 250 years, and this paper presents the first solution. Identifying a chain complex of topological vector spaces with a rich algebra of bounded operators gave us a new approach to tackle the problem.

The first solutions to Plateau’s problem by Douglas [Dou31], for which he won the first Field’s medal, were found by defining surfaces as parametrized images of a disk, and thus did not permit nonorientable surfaces or triple junctions. Douglas used the integral of the Jacobian of the parametrizing map to define area. Figure 2 shows that the “classical solutions” of Douglas can have transverse self-intersections which are never seen in soap films. Osserman [Oss70], Alt [Alt73], and Gulliver [Gul73] proved any classical solution of Douglas must be an immersion of a disk. The solutions of Federer and Fleming [FF60] using surfaces defined as integral currents are necessarily orientable. They define area using the mass norm of a current. They were awarded the Steele prize “for their pioneering work in Normal and Integral currents” [FF60]. Fleming [Fle62] proved such solutions are smoothly embedded away from γ , and regularity near a smooth boundary (5) was later established in [HS79]. Plateau’s problem remained an open problem because none of the solutions given by Douglas or Federer and Fleming permits the Möbius strip as the solution (b) for the curve in Figure 1. Instead, their methods produce solution (a), an oriented embedded disk.

¹Plateau’s original experiments [Pla73] were with oil which does not form films across wire frames as easily as soap/glycerin solutions.

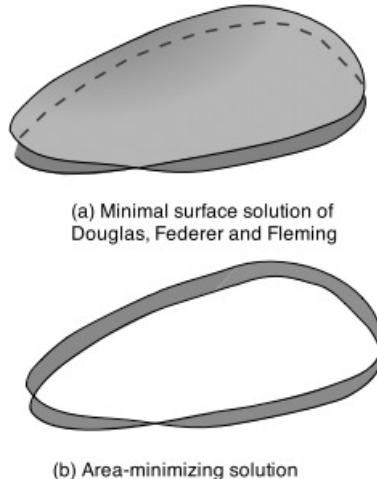


Figure 1: The Möbius strip

Reifenberg [Rei60] used point sets to define his surfaces and Hausdorff measure for area, but did not have a boundary operator. His surfaces could model nonorientable examples, but not those with triple junctions. Fleming and Ziemer’s flat chains (mod 2) [Fle66, Zie62] contain Möbius strips, but no surfaces with triple junctions. Fleming’s flat chains (mod 3) permit triple junctions but not Möbius strips.

Almgren’s integral varifolds [Alm66] provide models for all soap films. A 2-varifold is defined as a Radon measure on the product of \mathbb{R}^3 with the Grassmannian of 2-planes through the origin of \mathbb{R}^3 . He proved a compactness theorem for integral varifolds with bounds on areas, first variations and supports. For a time, there was excited optimism about Almgren’s methods. Ziemer’s Bulletin review [Zie69] referred to varifolds as “a new and promising approach to the old and formidable Plateau’s problem.” However, the lack of a boundary operator on varifolds (see, for example, [Mor88], §11.2), has made the proof of existence of an area minimizer given by a compactness theorem for varifolds elusive. The problem remained open with most mathematicians not realizing it.

In this paper, we answer (1)-(3) in the affirmative. Our solutions are geometrically meaningful, not just weak solutions, since all differential k -chains are approximated by “Dirac k -chains” which we define as formal sums $\sum_{i=1}^m (p_i; \alpha_i)$ where $p_i \in M$ and $\alpha_i \in \Lambda_k(T_{p_i}(M))$. Dirac chains have a natural and simple constructive geometric description (see §2.4.1). Our methods extend to a number of other variational problems and to k -dimensional cycles in Riemannian n -manifolds M

for all $0 \leq k \leq n - 1$.

We roughly state our main theorem for Jordan curves in \mathbb{R}^3 before defining all of the terms.

Theorem 1.0.1. *Given a smooth Jordan curve γ in \mathbb{R}^3 , there exists a surface S_0 spanning γ with minimal area where S_0 is an element of a certain topological vector space which includes representatives of all types of observed soap films as well as all smoothly immersed surfaces of all genus types, orientable or nonorientable, including those with possibly multiple junctions.*

See Theorem 7.3.3 for a precise statement. Our compactness Theorem 7.3.2 leading to Theorem 1.0.1 is the first compactness theorem in an infinite dimensional space taking into account all known soap films and smoothly immersed surfaces, and for which the boundary operator is well-defined, continuous, and maps the solution to a “representative” $\tilde{\gamma}$ of the prescribed curve γ . Figure 8 shows that γ does not have to be a closed curve². However, $\partial\tilde{\gamma} = 0$, even for the example in Figure 8 without the need for “hidden wires”.

Figures 1 and 2 demonstrate the differences between solutions found by Douglas, Federer, Fleming, and the author for the Möbius strip and a simple modification of it. It is of interest to settle what

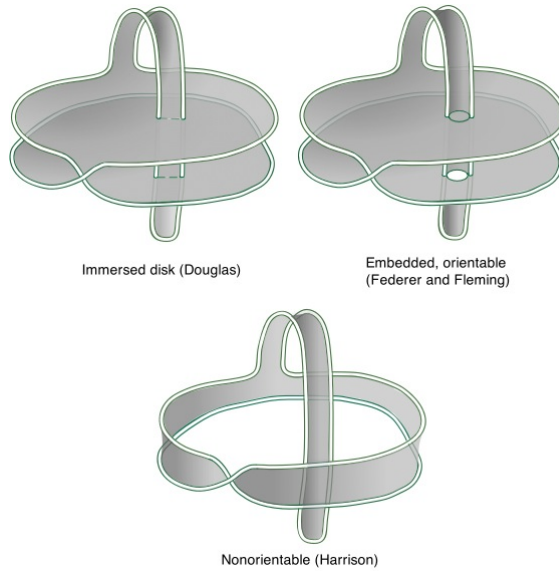


Figure 2: Three different solutions from three different viewpoints

any singularities must look like in order to solve question (4) – soap film regularity. This is not the goal of this paper. However, regularity appears to in sight, using Almgren’s $(M, 0, \delta)$ -minimizing

²This example was used by Almgren to defend the lack of a boundary operator for varifolds as natural

sets, the regularity results of Taylor [Tay76] showing that a $(M, 0, \delta)$ -minimal set has the soap film structure that Plateau had observed a hundred years earlier, and a generalization of Taylor's result which was recently published by David [Dav10] and relies on a variant of Reifenberg's topological disk theorem [Rei60].

In our earlier papers [Har04a, Har04b] we found a solution to Plateau's soap film problem, assuming a bound on the total length of triple junctions, an artificial condition we have now discarded using the new methods of in [Har10].

The author thanks Morris Hirsch and Harrison Pugh for their many helpful comments, questions, and insights. Steven Krantz has supported research of numerous mathematicians working on aspects of Plateau's problem through his superb editorial work for the Journal of Geometric Analysis. This paper would not exist had it not been for his early interest and encouragement. She wishes to thank Frank Morgan [Mor88] for his excellent course on geometric measure at the Mathematical Sciences Research Institute in 2001 where he informed the audience that the general problem of Plateau was still open.

2 Differential chains of type B

This work relies on methods of calculus presented in [Har10]. In this preliminary section we recount the definition of the bigraded chain complex of topological vector spaces $\hat{\mathcal{B}}_k^r(U)$ and those operators from [Har10] which we use in this paper.

2.1 Dirac chains

For U open in \mathbb{R}^n , let $\mathcal{A}_k = \mathcal{A}_k(U)$ be the free vector space of *Dirac k -chains in U* , i.e., finitely supported functions $U \rightarrow \Lambda_k(\mathbb{R}^n)$, expressed in the formal sum notation $\sum(p_i; \alpha_i)$ where $p_i \in U$ and $\alpha_i \in \Lambda_k(\mathbb{R}^n)$. (We use the standard convention of formal sums in which the only relations permitted are when the base points are the same. For example, $(p; \alpha) + (p; \beta) = (p; \alpha + \beta)$ and $2(p; \alpha) = (p; 2\alpha)$.) We call $(p; \alpha)$ a *k -element in U* if $\alpha \in \Lambda_k(\mathbb{R}^n)$ and $p \in U$. If α is simple, then $(p; \alpha)$ a *simple k -element in U* .

2.2 Mass norm

An inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^n determines the mass norm on $\Lambda_k(U)$ as follows: Let $\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det(\langle u_i, v_j \rangle)$. The *mass* of a simple k -vector $\alpha = v_1 \wedge \cdots \wedge v_k$ is defined by $\|\alpha\| := \sqrt{\langle \alpha, \alpha \rangle}$.

The mass of a k -vector α is $\|\alpha\| := \inf \left\{ \sum_{j=1}^N \|(\alpha_i)\| : \alpha_i \text{ are simple, } \alpha = \sum_{j=1}^N \alpha_i \right\}$. Define the *mass* of a k -element $(p; \alpha)$ by $\|(p; \alpha)\|_{B^0} := \|\alpha\|$. Mass is a norm on the subspace of Dirac k -chains supported in p , since that subspace is isomorphic to the exterior algebra $\Lambda(\mathbb{R}^n) = \bigoplus_{k=0}^n \Lambda_k(\mathbb{R}^n)$ for which mass is a norm (see [Fed69], p 38-39). The *mass* of a Dirac k -chain $A = \sum_{i=1}^m (p_i; \alpha_i) \in \mathcal{A}_k(U)$ is given by

$$\|A\|_{B^0} := \sum_{i=1}^m \|(p_i; \alpha_i)\|_{B^0}.$$

If a different inner product is chosen, the resulting masses of Dirac chains are topologically equivalent. It is straightforward to show that $\|\cdot\|_{B^0}$ is a norm on $\mathcal{A}_k(U)$.

2.3 Difference chains and the B^r norm

Given a k -element $(p; \alpha)$ with $p \in U$ and $u \in \mathbb{R}^n$, let $T_u(p; \alpha) := (p + u; \alpha)$ be translation through u , and $\Delta_u(p; \alpha) := (T_u - I)(p; \alpha)$. Let $S^j = S^j(\mathbb{R}^n)$ be the j -th *symmetric power* of the symmetric algebra $S(\mathbb{R}^n)$. Denote the symmetric product in the symmetric algebra $S(\mathbb{R}^n)$ by \circ . Let $\sigma = \sigma^j = u_1 \circ \cdots \circ u_j \in S^j$ with $u_i \in \mathbb{R}^n, i = 1, \dots, j$. Recursively define $\Delta_{u \circ \sigma^j}(p; \alpha) := (T_u - I)(\Delta_{\sigma^j}(p; \alpha))$. Let $\|\sigma\| := \|u_1\| \cdots \|u_j\|$ and $|\Delta_{\sigma^j}(p; \alpha)|_{B^j} := \|\sigma\| \|\alpha\|$. Let $\Delta_{\sigma^0}(p; \alpha) := (p; \alpha)$, to keep the notation consistent. We say $\Delta_{\sigma^j}(p; \alpha)$ is *inside* U if the convex hull of $\text{supp}(\Delta_{\sigma^j}(p; \alpha))$ is a subset of U .

Definition 2.3.1. For $A \in \mathcal{A}_k(U)$ and $r \geq 0$, define the *seminorm*

$$\|A\|_{B^r, U} := \inf \left\{ \sum_{j=0}^r \sum_{i=1}^{m_j} \|\sigma_{j_i}\| \|\alpha_{j_i}\| : A = \sum_{j=0}^r \sum_{i=1}^{m_j} \Delta_{\sigma_{j_i}^j}(p_{j_i}; \alpha_{j_i}) \text{ where } \Delta_{\sigma_{j_i}^j}(p_{j_i}; \alpha_{j_i}) \text{ is inside } U \right\}.$$

For simplicity, we often write $\|A\|_{B^r} = \|A\|_{B^r, U}$ if U is understood. It is easy to see that the B^r norms on Dirac chains are decreasing as r increases.

It is shown in [Har10] (Theorems 2.6.1 and 6.4.2) that $\|\cdot\|_{B^r}$ is a norm on the free space of Dirac k -chains \mathcal{A}_k called the B^r norm. Let $\hat{\mathcal{B}}_k^r = \hat{\mathcal{B}}_k^r(U)$ be the Banach space obtained upon completion of $\mathcal{A}_k(U)$ with the B^r norm. Elements of $\hat{\mathcal{B}}_k^r(U)$, $0 \leq r \leq \infty$, are called *differential k -chains of class B^r in U* . The natural inclusions $\hat{\mathcal{B}}_k^r(U_1) \hookrightarrow \hat{\mathcal{B}}_k^r(U_2)$ are continuous for all open $U_1 \subset U_2 \subset \mathbb{R}^n$. In [Har10] we also study the inductive limit $\mathcal{B}_k = \mathcal{B}_k(U) := \varinjlim \hat{\mathcal{B}}_k^r(U)$ as $r \rightarrow \infty$, endowed with the inductive limit topology, obtaining a *DF-space*. However, for this paper it suffices to work within a subcomplex of the simpler bigraded chain complex of Banach spaces $\hat{\mathcal{B}}_k^r$.

Let $\mathcal{B}_k^0(U)$ be the Banach space of bounded and measurable k -forms, $\mathcal{B}_k^1(U)$ the Banach space of bounded Lipschitz k -forms, and for each $r > 1$, let $\mathcal{B}_k^r(U)$ be the Banach space of differential k -forms, each with uniform bounds on each of the s -th order directional derivatives for $0 \leq s \leq r - 1$

and the $(r-1)$ -st derivatives satisfy a bounded Lipschitz condition. Denote the resulting norm by $\|\omega\|_{B^r} = \sup_{|\ell| \leq r-1} \{\|D^\ell \omega\|_{\text{sup}}, \|D^\ell \omega\|_{\text{Lip}}\}$. We always denote differential forms by lower case Greek letters such as ω, η and differential chains by upper case Roman letters such as J, K, A , so there is no confusion when we write $\|\omega\|_{B^r}$ or $\|J\|_{B^r}$. Elements of $\mathcal{B}_k^r = \mathcal{B}_k^r(U)$ are called *differential k -forms of class B^r in U* .

Theorem 2.3.2 (Isomorphism Theorem). $\hat{\mathcal{B}}_k^r(U)' \cong \mathcal{B}_k^r(U)$ and the integral pairing $\int : \hat{\mathcal{B}}_k^r(U) \times \mathcal{B}_k^r(U) \rightarrow \mathbb{R}$ where $(J, \omega) \mapsto \omega(J)$ is bilinear, nondegenerate, and jointly continuous.

(See [Har10] Theorem 6.4.3)

Define

$$\int_J \omega := \omega(J)$$

for all $J \in \hat{\mathcal{B}}_k^r(U)$ and $\omega \in \mathcal{B}_k^r(U)$.

Definition 2.3.3. If $J \in \hat{\mathcal{B}}_k^r(U)$ is nonzero then its support $\text{supp}(J)$ is the smallest closed subset $E \subset U$ such that $\int_J \omega = 0$ for all smooth ω with compact support disjoint from E . Support of a nonzero differential chain is a uniquely determined nonempty set (see [Har10] Theorems 5.1.3 and 5.2.2).

2.4 Pushforward

Suppose $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^m$ are open and $F : U_1 \rightarrow U_2$ is a differentiable map. For $p \in U_1$ and $v_1 \wedge \cdots \wedge v_k \in \Lambda_k(\mathbb{R}^n)$, define *linear pushforward* $F_{p*}(v_1 \wedge \cdots \wedge v_k) := DF_p(v_1) \wedge \cdots \wedge DF_p(v_k)$ where DF_p is the total derivative of F at p . Define $F_*(p; \alpha) := (F(p), F_{p*}\alpha)$ for all simple k -elements $(p; \alpha)$ and extend to a linear map $F_* : \mathcal{A}_k(U_1) \rightarrow \mathcal{A}_k(U_2)$ called *pushforward*. Define $F^*\omega := \omega F_*$ for exterior k -forms $\omega \in \mathcal{A}_k(U)^*$. Then F^* is the classical pullback $F^* : \mathcal{A}_k(U_2)^* \rightarrow \mathcal{A}_k(U_1)^*$.

Definition 2.4.1.

Let $\mathcal{M}^r(U, \mathbb{R}^m)$ be the vector space of differentiable maps $F : U \rightarrow \mathbb{R}^m$ so that the directional derivatives $L_{e_j} F_i$ of its coordinate functions F_i are of class B^{r-1} , for $r \geq 1$. Define the seminorm $|F|_{D^r, U} := \max_{i,j} \{\|L_{e_j} F_i\|_{B^{r-1}, U}\}$. We write $|F|_{D^r} = |F|_{D^r, U}$ when U is understood. Let $\mathcal{M}^r(U_1, U_2) := \{F \in \mathcal{M}^r(U_1, \mathbb{R}^m) : F(U_1) \subset U_2 \subset \mathbb{R}^m\}$.

A map $F \in \mathcal{M}^1(U, \mathbb{R}^m)$ may not be bounded, but its directional derivatives must be. An important example is the identity map $x \mapsto x$ which is an element of $\mathcal{M}^1(\mathbb{R}^n, \mathbb{R}^n)$.

Theorem 2.4.2. *If $F \in \mathcal{M}^r(U_1, U_2)$, then F_* satisfies*

$$\|F_*(A)\|_{B^r, U_2} \leq \max\{1, r\|F\|_{D^r, U_1}\}\|A\|_{B^r, U_1}$$

for all $A \in \mathcal{A}_0(U_1)$ and $r \geq 0$. It follows that $F_ : \hat{\mathcal{B}}_k^r(U_1) \rightarrow \hat{\mathcal{B}}_k^r(U_2)$ and $F^* : \mathcal{B}_k^r(U_2) \rightarrow \mathcal{B}_k^r(U_1)$ are continuous bigraded operators with $\int_{F_*J} \omega = \int_J F^* \omega$ for all $J \in \hat{\mathcal{B}}_k^r(U_1)$ and $\omega \in \mathcal{B}_k^r(U_2)$. Furthermore, $\partial \circ F_* = F_* \circ \partial$.*

(See Theorem 6.5.4 in [Har10].)

2.4.1 Algebraic chains

Theorem 2.4.3. *[Representatives of k -cells] Each oriented affine k -cell σ in U naturally corresponds to a unique differential k -chain $\tilde{\sigma} \in \hat{\mathcal{B}}_k^1(U)$ in the sense of integration of differential forms. That is, the Riemann integral and the differential chain integral coincide for all Lipschitz k -forms ω :*

$$\int_{\sigma} \omega = \int_{\tilde{\sigma}} \omega.$$

The proof of this theorem may be found in [Har10] (Theorem 2.10.2). It follows that each bounded open set U in \mathbb{R}^n with the standard orientation of \mathbb{R}^n is uniquely represented by an n -chain $\tilde{U} \in \hat{\mathcal{B}}_n^1(U)$. We define *polyhedral chains* $\sum_{i=1}^s a_i \tilde{\sigma}_i$, $a_i \in \mathbb{R}$ and σ_i an oriented affine k -cell. This coincides with the classical definition as found in [Whi57]. Polyhedral chains are dense in $\hat{\mathcal{B}}_k^r(U)$ (see [Har10] Theorem 2.11.6).

If σ is an oriented affine k -cell in U and $F \in M^1(U, W)$, then $F_* \tilde{\sigma} \in \hat{\mathcal{B}}_k^1(W)$, and is called an *algebraic k -cell*. We remark that an algebraic k -cell $F_* \tilde{\sigma}$ is not the same as a singular k -cell $F\sigma$ from algebraic topology. For example, if $F(x) = x^2$ and $\sigma = (-1, 1)$, then the algebraic 1-cell $F_* \tilde{\sigma} = 0$, but the singular 1-cell $F\sigma \neq 0$.

If $(p; \alpha)$ is a simple k -element, we may write $(p; \alpha) = \lim_{i \rightarrow \infty} 2^{ki-1} Q_i(p)$ where $Q_1(p)$ is an oriented affine k -cell in the k -direction of α containing p with unit diameter, unit k -volume, and $Q_i(p)$ is a homothetic replica of $Q_1(p)$, containing p , and with diameter 2^{-i} (see [Har10] Lemma 2.11.5). This gives us the promised geometric interpretation of the simple k -element $(p; \alpha)$ as a k -dimensional point mass, a limit of shrinking renormalized oriented affine k -cells. (We can also use any sequence of limiting chains as long as their supports tend to p , their k -directions are the same and the masses tend to $\|\alpha\|_0$. There is nothing special about squares, except for computational convenience.)

2.5 Vector fields

Let $\mathcal{V}^r(U)$ be the vector space of vector fields X on U whose local coordinate functions ϕ_i are of class B^r . In particular, if $r = 0$, then the time- t map of the flow of X is Lipschitz. For $X \in \mathcal{V}^r(U)$ define $\|X\|_{B^r} = \max\{\|\phi_i\|_{B^r}\}$. Then $\|\cdot\|_{B^r}$ is a norm on $\mathcal{V}^r(U)$. We say that X is of class B^r if $X \in \mathcal{V}^r(U)$.

3 Operators

3.1 Extrusion

Let $X \in \mathcal{V}^r(U)$. Define the graded operator *extrusion* $E_X : \mathcal{A}_k(U) \rightarrow \mathcal{A}_{k+1}(U)$ by $E_X(p; \alpha) := (p; X(p) \wedge \alpha)$ for all $p \in U$ and $\alpha \in \Lambda_k(\mathbb{R}^n)$. Then $i_X \omega := \omega E_X$ is the classical interior product $i_X : \mathcal{A}_{k+1}(U)^* \rightarrow \mathcal{A}_k(U)^*$.

Theorem 3.1.1. *If $X \in \mathcal{V}^r(U)$ and $A \in \mathcal{A}_k(U)$, then*

$$\|E_X(A)\|_{B^r} \leq n^2 r \|X\|_{B^r} \|A\|_{B^r}.$$

(For a proof see [Har10], Theorem 8.2.2.)

Therefore, E_X extends to a continuous operator $E_X : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k+1}^r(U)$. It follows from the isomorphism theorem 2.3.2 that

$$\int_{E_X J} \omega = \int_J i_X \omega. \quad (3.1)$$

Only a few operators we work with are closed and bounded on Dirac chains such as extrusion E_X and pushforward F_* .

3.2 Retraction

For $\alpha = v_1 \wedge \cdots \wedge v_k \in \Lambda_k(\mathbb{R}^n)$, let $\hat{\alpha}_i := v_1 \wedge \cdots \wedge \hat{v}_i \cdots \wedge v_k \in \Lambda_{k-1}(\mathbb{R}^n)$. For $X \in \mathcal{V}^r(U)$ define the graded operator *retraction* $E_X^\dagger : \mathcal{A}_k(U) \rightarrow \mathcal{A}_{k-1}(U)$ by $(p; \alpha) \mapsto \sum_{i=1}^k (-1)^{i+1} \langle X(p), v_i \rangle (p; \hat{\alpha}_i)$, for $p \in U$. A straightforward calculation shows this to be the adjoint of wedge product with $X(p)$ at a point p , and thus is well-defined. The dual operator on forms is wedge product with the 1-form X^\flat representing the vector field X via the inner product with $X^\flat \wedge \cdot : \mathcal{A}_{k-1}(U)^* \rightarrow \mathcal{A}_k(U)^*$.

Theorem 3.2.1. *If X is a vector field on U of class B^r and $J \in \hat{\mathcal{B}}_k^r(U)$, then*

$$\|E_X^\dagger(J)\|_{B^r} \leq k n^2 r \|X\|_{B^r} \|J\|_{B^r}.$$

It follows that $E_X^\dagger : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k-1}^r(U)$ and $X^\flat \wedge \cdot : \mathcal{B}_{k-1}^r(U) \rightarrow \mathcal{B}_k^r(U)$ are continuous graded operators with

$$\int_{E_X^\dagger J} \omega = \int_J X^\flat \wedge \omega \quad (3.2)$$

for all $J \in \hat{\mathcal{B}}_k^{r+1}(U)$ and $\omega \in \mathcal{B}_k^r(U)$ (see [Har10] Theorem 9.2.2)

3.3 Boundary

There are several equivalent ways to define the boundary operator $\partial : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k-1}^{r+1}(U)$. We have found it very useful to define boundary on Dirac chains directly. For $v \in \mathbb{R}^n$, and a simple k -element $(p; \alpha)$ with $p \in U$, let $P_v(p; \alpha) := \lim_{t \rightarrow 0} (p + tv; \alpha/t) - (p; \alpha/t)$. It is perhaps surprising that this limit is nonzero if $\alpha \neq 0$. It is shown in [Har10] (Lemma 3.3.1) that this limit exists as a well-defined element of $\hat{\mathcal{B}}_k^2(U)$. We may then linearly extend $P_v : \mathcal{A}_k(U) \rightarrow \hat{\mathcal{B}}_k^1(U)$. Moreover, $\|P_v(A)\|_{B^{r+1}} \leq \|v\| \|A\|_{B^r}$ for all $A \in \mathcal{A}_k(U)$ (see [Har10] Lemma 3.3.2). For an orthonormal basis $\{e_i\}$ of \mathbb{R}^n , set $\partial := \sum P_{e_i} E_{e_i}^\dagger$. Since P_{e_i} and E_{e_i} are continuous, ∂ is a well-defined continuous operator $\partial : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k-1}^{r+1}(U)$ that restricts to the classical boundary operator on polyhedral k -chains independent of choice of $\{e_i\}$ (see [Har10] (Corollary 3.5.2 and Lemma 3.5.5)).

Theorem 3.3.1. *[General Stokes' Theorem] The bigraded operator boundary $\partial : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k-1}^{r+1}(U)$ is continuous with $\partial \circ \partial = 0$, and $\|\partial J\|_{B^{r+1}} \leq kn \|J\|_{B^r}$ for all $J \in \hat{\mathcal{B}}_k^r$ and $r \geq 0$. Furthermore, if $\omega \in \mathcal{B}_{k-1}^r(U)$ is a differential form and $J \in \hat{\mathcal{B}}_k^{r-1}(U)$ is a differential chain, then*

$$\int_{\partial J} \omega = \int_J d\omega.$$

The proof of this may be found in [Har10] (Theorems 3.5.1 and 3.5.4).

We say a differential k -chain $J \in \mathcal{B}_k(U)$ is a *differential k -cycle in U* if $\partial J = 0$.

3.4 Prederivative

Definition 3.4.1. *Suppose $X \in \mathcal{V}^r(U)$. Define the linear map prederivative $P_X : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^{r+1}(U)$ by*

$$P_X := E_X \partial + \partial E_X.$$

This agrees with the previous definition of P_v for $v \in \mathbb{R}^n$ in §3.3 since $E_v \partial + \partial E_v = \sum_i P_{e_i} (E_v E_{e_i}^\dagger - E_{e_i}^\dagger E_v) = \sum_i P_{e_i} \langle v, e_i \rangle I = P_v$.

It follows from Theorems 3.1.1 and 3.3.1 that both E_X and ∂ are continuous. Therefore, P_X is continuous. Its dual operator L_X is the classically defined Lie derivative since $L_X = i_X d + di_X$ by Theorems 3.1.1 and 3.3.1, and this uniquely determines L_X . It follows that $P_X : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^{r+1}(U)$ and Lie derivative $L_X : \mathcal{B}_k^r(U) \rightarrow \mathcal{B}_k^{r-1}(U)$ are continuous bigraded linear operators. By the isomorphism theorem 2.3.2, this implies the duality relation:

$$\int_{P_X J} \omega = \int_J L_X \omega \quad (3.3)$$

for all $J \in \hat{\mathcal{B}}_k^{r-1}(U)$ and $\omega \in \mathcal{B}_k^r(U)$. Furthermore, $P_X \partial = \partial P_X$ since $P_X = \partial E_X + E_X \partial$ implies $\partial E_X = \partial E_X \partial = P_X \partial$. We remark that $P_u \circ P_v = P_v \circ P_u$ for fixed $u, v \in \mathbb{R}^n$, but for non-constant vector fields X, Y , the operators P_X, P_Y do not necessarily commute³.

Theorem 3.4.2. *If $X \in \mathcal{V}^r(U)$, then prederivative*

$$P_X : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_k^{r+1}(U)$$

satisfies

$$\|P_X(J)\|_{B^{r+1}} \leq 2kn^3 r \|X\|_{B^r} \|J\|_{B^r}$$

for all $J \in \hat{\mathcal{B}}_k^r(U)$ and $0 \leq r < \infty$.

(See Theorem 8.4.1 of [Har10])

Prederivative gives us a way to “geometrically differentiate” a differential chain in the infinitesimal directions determined by a vector field, even when the support of the differential chain is highly nonsmooth, and without using any functions or forms.

Theorem 3.4.3. *If $X \in \mathcal{V}^r(U)$ and $(p; \alpha)$ is a k -element with $p \in U$, then $P_X(p; \alpha) \in \hat{\mathcal{B}}_k^1(U)$ with*

$$P_X(p; \alpha) = \lim_{t \rightarrow 0} (\phi_t(p); \phi_{t*} \alpha / t) - (p; \alpha / t)$$

where ϕ_t is the time- t map of the flow of X .

The proof of this may be found in [Har10] (Theorem 8.4.3).

³The universal enveloping algebra can be used instead of the symmetric algebra.

3.4.1 The chainlet complex

If $\tau = u_s \circ \cdots \circ u_1 \in S^s(\mathbb{R}^n)$, the s -th order symmetric power of the symmetric algebra, let $P_\tau := P_{u_s} \circ \cdots \circ P_{u_1}$.

Definition 3.4.4. For U open in \mathbb{R}^n , let $\mathcal{A}_k^s(U) := \{\sum P_{\tau_i}(p_i; \alpha_i) \mid p_i \in U, \tau_i \in S^s(\mathbb{R}^n), \text{ and } \alpha_i \in \Lambda_k(\mathbb{R}^n)\}$. Elements of $\mathcal{A}_k^s(U)$ are called Dirac k -chains of dipole order s in U . Let

$$\mathcal{Ch}_k^s(U) := \overline{(\mathcal{A}_k^s(U), \|\cdot\|_{B^{s+1}})}.$$

Elements of $\mathcal{Ch}_k^s(U)$ are called k -chainlets of dipole order s . For $s \geq 1$, the space $\mathcal{Ch}_k^s(U)$ is a strict topological subspace of the Banach space $\hat{\mathcal{B}}_k^{s+1}(U)$, while $\mathcal{Ch}_k^0(U) = \hat{\mathcal{B}}_k^1(U)$.

In this paper, we only need $0 \leq s \leq 1$ and $n-1 \leq k \leq n$, but include the entire chainlet complex here for completion.

It is not hard to see that the primitive operators E_V, E_V^\dagger and P_V , as well as the operators pushforward and multiplication by a function, are continuous and closed on the direct sum $\oplus_k \oplus_s \mathcal{Ch}_k^s(U)$. It follows that boundary is continuous, and thus $\mathcal{Ch}_k^s(U)$ is a bigraded topological chain complex and is a proper subcomplex of the differential chain complex $\hat{\mathcal{B}}_k^{s+1}(U)$ with the induced topology.

$$\begin{array}{ccccccc} \hat{\mathcal{B}}_n^{s+1} & \xrightarrow{\partial} & \hat{\mathcal{B}}_{n-1}^{s+2} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \hat{\mathcal{B}}_1^{n+s} & \xrightarrow{\partial} & \hat{\mathcal{B}}_0^{n+s+1} \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ \mathcal{Ch}_n^s & \xrightarrow{\partial} & \mathcal{Ch}_{n-1}^{s+1} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \mathcal{Ch}_1^{n+s-1} & \xrightarrow{\partial} & \mathcal{Ch}_0^{n+s} \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ \mathcal{A}_n^s & \xrightarrow{\partial} & \mathcal{A}_{n-1}^{s+1} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \mathcal{A}_1^{n+s-1} & \xrightarrow{\partial} & \mathcal{A}_0^{n+s} \end{array}$$

In §7 we work almost entirely in the chainlet complex, but not until then.

4 Integral monopole and dipole chains

If X is a Lipschitz vector field defined in a neighborhood of a smoothly embedded k -cell $\tau \subset U$ and whose component to τ is unit, then $E_X \tilde{\tau} \in \mathcal{Ch}_{k+1}^0(U)$ is an *integral monopole k -cell* and $P_X \tilde{\tau} \in \mathcal{Ch}_k^1(U)$ is an *integral dipole k -cell* in U . Finite sums $\sum_{i=1}^m E_{X_i} \tilde{\tau}_i$ and $\sum_{i=1}^m P_{X_i} \tilde{\tau}_i$ are *integral monopole and dipole k -chains* in U , respectively. The group of integral dipole k -chains in U is denoted $\mathcal{I}_k^1(U) \subset \mathcal{Ch}_k^1(U)$. Chains of the form $\bar{S} = \sum_{i=1}^m E_{X_i} \tilde{\tau}_i \in \mathcal{Ch}_{k+1}^0(U)$ are *integral monopole*

k-chains. The group of integral monopole *k*-chains is $\mathcal{I}_k^0(U) \subset \mathcal{Ch}_{k+1}^0(U)$. (In this paper we only consider $k = 1$ or $k = 2$ where U is open in \mathbb{R}^3 .) The completion $\overline{\mathcal{I}_k^1(U)}$ of the subspace generated by integral dipole *k*-chains uses the B^2 norm, and $\overline{\mathcal{I}_k^0(U)}$ uses the B^1 -norm. Then $\mathcal{I}_k^s(U) \subset \overline{\mathcal{I}_k^s(U)} \subset \mathcal{Ch}_k^s(U) \subset \hat{\mathcal{B}}_k^{s+1}$ for $s = 0, 1$.

4.1 Volume functional for computing area of integral dipole chains

Definition 4.1.1. *The area of an integral dipole $(n - 1)$ -cell $P_X \tilde{\tau}$ is defined by*

$$A(P_X \tilde{\tau}) := \int_{E_X \tilde{\tau}} dV$$

where dV is the volume form in \mathbb{R}^n . We say $P_X \tilde{\tau}$ is positively oriented if $A(P_X \tilde{\tau}) \geq 0$. Henceforth, we assume that all integral dipole $(n - 1)$ -cells are positively oriented. If $S = \sum_{i=1}^s P_{X_i} \tilde{\tau}_i$ is an integral dipole $(n - 1)$ -chain, define

$$A(S) := \sum_{i=1}^s A(P_{X_i} \tilde{\tau}_i).$$

Since the orientations are all positive, $A(S) \geq 0$ and takes into account multiplicity where cells overlap.

All of the surfaces in the figures provide examples of integral monopole and dipole surfaces, as do all soap films observed in nature with the simple structure observed by Plateau, as well as all smooth images of disks, embedded orientable surfaces, nonorientable surfaces, and surfaces with multiple junctions. The boundary of an integral dipole surface is a well-defined dipole curve, and will also be integral in our constructions below. Lipid bilayers and physical soap films are naturally modeled by integral dipole surfaces because of the hydrophobic effect⁴.

- The torus-annulus Figure 6 can be represented by a integral dipole surface by creating a triple junction along the curve in the dipole torus where it meets the dipole annulus.

⁴ “Natural bilayers are usually made mostly of phospholipids, which have a hydrophilic head and two hydrophobic tails. When phospholipids are exposed to water, they arrange themselves into a two-layered sheet (a bilayer) with all of their tails pointing toward the center of the sheet. The center of this bilayer contains almost no water and also excludes molecules like sugars or salts that dissolve in water but not in oil. This assembly process is similar to the coalescing of oil droplets in water and is driven by the same force, called the hydrophobic effect. Because lipid bilayers are quite fragile and are so thin that they are invisible in a traditional microscope, bilayers are very challenging to study” http://en.wikipedia.org/wiki/Lipid_bilayer

- The Möbius strip μ becomes orientable in its dipole version as seen in the third drawing of Figure 2.
- The “Y-problem” mathematicians have faced is the following: Consider the chain $\sum_{i=1}^3 \tilde{\tau}_i$ of three oriented affine cells meeting only along a mutual edge L at 120 degrees as in Figure 3. Then $\text{supp}(\partial \sum_{i=1}^3 \tilde{\tau}_i) \cap L = L$. This makes the boundary operator not very useful when dealing with triple junctions of cellular chains. However, if we replace each $\tilde{\tau}_i$ with $P_{X_i} \tilde{\tau}_i$, then $\text{supp}(\partial \sum P_{X_i} \tilde{\tau}_i) \cap L$ is just the union of the two endpoints of L .
- By allowing non-orthogonal vector fields X in our definition of integral dipole cells, we may construct models for multiple junctions of a surface meeting in arbitrary angles.

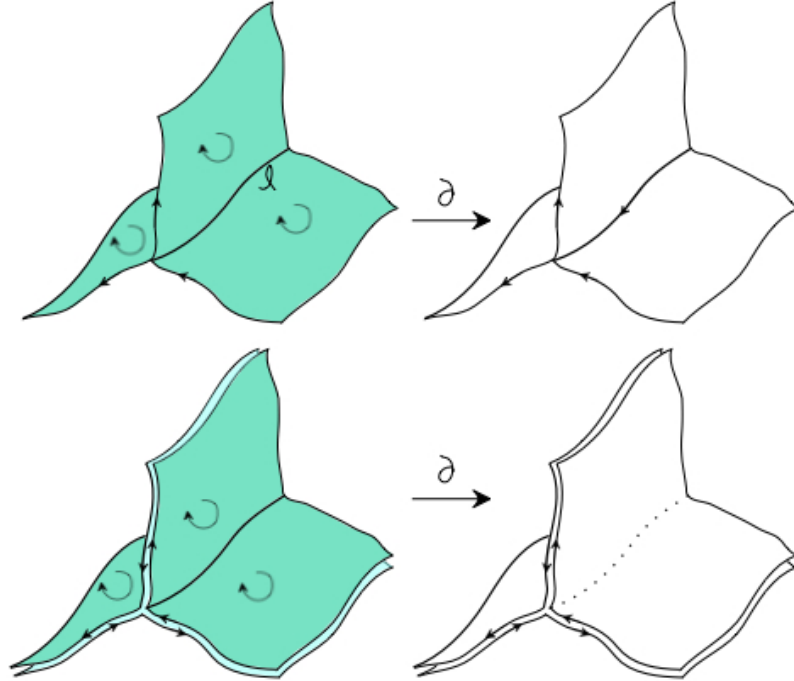


Figure 3: The Y-problem resolved

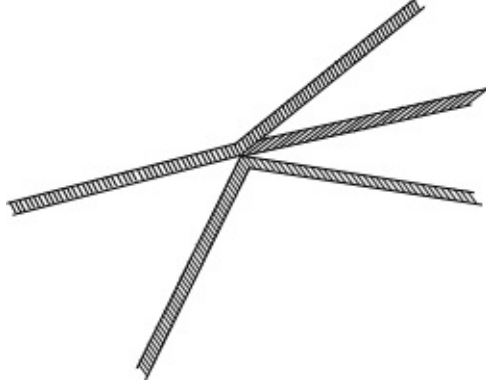


Figure 4: Multiple branches

5 Geometric Poincaré Lemma

5.1 Cartesian wedge product

Suppose $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}^m$ are open. Let $\iota_1 : U_1 \rightarrow U_1 \times U_2$ and $\iota_2 : U_2 \rightarrow U_1 \times U_2$ be the inclusions $\iota_1(p) = (p, 0)$ and $\iota_2(q) = (0, q)$. Let $\pi_1 : U_1 \times U_2 \rightarrow U_1$ and $\pi_2 : U_1 \times U_2 \rightarrow U_2$ be the projections $\pi_i(p_1, p_2) = p_i$, $i = 1, 2$. Let $(p; \alpha) \in \mathcal{A}_k(U_1)$ and $(q; \beta) \in \mathcal{A}_\ell(U_2)$. Define $\times : \mathcal{A}_k(U_1) \times \mathcal{A}_\ell(U_2) \rightarrow \mathcal{A}_{k+\ell}(U_1 \times U_2)$ by

$$\times((p; \alpha), (q; \beta)) := ((p, q); \iota_{1*}\alpha \wedge \iota_{2*}\beta)$$

where $(p; \alpha)$ and $(q; \beta)$ are k - and ℓ -elements, respectively, and extend bilinearly. We call $P \times Q := \times(P, Q)$ the *Cartesian wedge product* of P and Q .

Theorem 5.1.1. *Cartesian wedge product $\times : \hat{\mathcal{B}}_k^r(U_1) \times \hat{\mathcal{B}}_\ell^s(U_2) \rightarrow \hat{\mathcal{B}}_{k+\ell}^{r+s}(U_1 \times U_2)$ is associative, bilinear and continuous for all open sets $U_1 \subseteq \mathbb{R}^n, U_2 \subseteq \mathbb{R}^m$ and satisfies*

1. $\|J \times K\|_{B^{r+s}, U_1 \times U_2} \leq \|J\|_{B^r, U_1} \|K\|_{B^s, U_2};$
2. $\|J \times \widetilde{(a, b)}\|_{B^r, U_1 \times \mathbb{R}} \leq |b - a| \|J\|_{B^r, U_1}$ where $\widetilde{(a, b)}$ is a 1-chain representing the interval (a, b) .
- 3.

$$\partial(J \times K) = \begin{cases} (\partial J) \times K + (-1)^k J \times (\partial K), & k > 0, \ell > 0 \\ (\partial J) \times K, & k > 0, \ell = 0 \\ J \times (\partial K), & k = 0, \ell > 0 \end{cases}$$

4. $J \times K = 0$ implies $J = 0$ or $K = 0$;
5. $(p; \sigma \otimes \alpha) \times (q; \tau \otimes \beta) = ((p, q); \sigma \circ \tau \otimes \iota_{1*}\alpha \wedge \iota_{2*}\beta)$;
6. $(\pi_{1*}\omega \wedge \pi_{2*}\eta)(J \times K) = \omega(J)\eta(K)$ for $\omega \in \mathcal{B}_k^r(U_1), \eta \in \mathcal{B}_\ell^s(U_2)$;
7. $\text{supp}(J \times K) = \text{supp}(J) \times \text{supp}(K)$.

See [Har10] (Proposition 9.1.2 and Theorem 9.1.3).

5.2 Geometric Poincaré Lemma

We say that U is *contractible* if there exists a map $F \in M^r([0, 1] \times U, U)$ and a point $p_0 \in U$ with $F(0, p) = p_0$ and $F(1, p) = p$ for all $p \in U$. According to Theorem 2.4.2 $F_* : \hat{\mathcal{B}}_{k+1}^r((0, 1) \times U) \rightarrow \hat{\mathcal{B}}_{k+1}^r(U)$ is continuous with

$$\|F_*((\widetilde{(0, 1)} \times A))\|_{B^r} \leq \max\{1, r\|F\|_{D^r}\}\|A\|_{B^r}.$$

Definition 5.2.1. Define the cone operator $\kappa : \hat{\mathcal{B}}_k^1(U) \rightarrow \hat{\mathcal{B}}_{k+1}^1(U)$ by $\kappa(A) := F_*((\widetilde{(0, 1)} \times A))$.

Theorem 5.2.2. $\kappa : \hat{\mathcal{B}}_k^r(U) \rightarrow \hat{\mathcal{B}}_{k+1}^r(U)$ is a continuous linear map satisfying $\|\kappa J\|_{B^r} \leq R\|J\|_{B^r}$ for all differential chains $J \in \hat{\mathcal{B}}_k^r(U)$, and for all $0 \leq k \leq n-1$ and $r \geq 1$. Furthermore $\kappa\partial + \partial\kappa = I$.

Proof. We know κ is continuous since pushforward and Cartesian wedge product are continuous. See Theorem 2.4.2 to establish the inequality.

Furthermore, by Theorem 5.1.1

$$\begin{aligned} \kappa\partial(p; \alpha) + \partial\kappa(p; \alpha) &= F_*((\widetilde{(0, 1)} \times \partial(p; \alpha)) + \partial F_*((\widetilde{(0, 1)} \times (p; \alpha))) \\ &= F_*((\widetilde{(0, 1)} \times \partial(p; \alpha)) + F_*\partial((\widetilde{(0, 1)} \times (p; \alpha))) \\ &= F_*((\widetilde{(0, 1)} \times \partial(p; \alpha)) + (F_*(\partial(\widetilde{(0, 1)} \times (p; \alpha)) - F_*((\widetilde{(0, 1)} \times \partial(p; \alpha)))) \\ &= F_*(\partial(\widetilde{(0, 1)} \times (p; \alpha))) \\ &= F_*((1; 1) \times (p; \alpha)) - F_*((0; 1) \times (p; \alpha)). \end{aligned}$$

Since $F(1, p) = p$ for all $p \in U$, we deduce $F_*((1; 1) \times (p; \alpha)) = (p; \alpha)$, and since $F(0, p) = p_0$, we obtain $F_*((0; 1) \times (p; \alpha)) = 0$. It follows that $\kappa\partial + \partial\kappa$ is the identity on Dirac chains, and thus is the identity operator on $\hat{\mathcal{B}}_k^r(U)$.

□

Corollary 5.2.3. *[Geometric Poincaré Lemma] Let $U_1 \subset U_2 \subset \mathbb{R}^n$ be open subsets where U_1 is B^r contractible in U_2 . If $J \in \hat{\mathcal{B}}_k^r(U_1)$ with $\partial J = 0$, then there exists $C \in \hat{\mathcal{B}}_{k+1}^r(U_2)$ with $\partial C = J$ for all $1 \leq k \leq n$ and $1 \leq r \leq \infty$.*

Proof. Let $C = \kappa J$. Note that C is unique up to addition of a chain boundary since $J = \partial(\kappa J + \partial C)$. \square

The classical Poincaré Lemma for differential forms of class \mathcal{B} follows since the homotopy operator H given by $H\omega := \omega\kappa$ is continuous. (One can also derive the classical formula $(H\omega)(p; \alpha) = \int_0^1 i_{\partial/\partial t} \omega(F_t(p); \alpha) dt$ directly from our definition of κ , and leave this for interested readers as an exercise.)

Corollary 5.2.4. *There does not exist a nonzero differential n -cycle $J \in \hat{\mathcal{B}}_n^r(U)$ supported in a contractible open set U of a smooth n -manifold M for all $n > 0$.*

Proof. If $\partial J = 0$, then $\kappa\partial J = 0$ since κ is a continuous operator. However, $\kappa J = 0$ since every $(n+1)$ -chain in \mathbb{R}^n is degenerate. Thus

$$J = \partial\kappa J + \kappa\partial J = 0.$$

\square

The next result is included for completion, and is not used in this paper:

Corollary 5.2.5 (General Intermediate Value Theorem). *Suppose $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a smooth map where $1 \leq n \leq m$, $J \in \hat{\mathcal{B}}_n^r(\mathbb{R}^m)$ and $K \in \hat{\mathcal{B}}_n^r(\mathbb{R}^n)$ with $\text{supp}(F_*J) \cup \text{supp}(K)$ a compact subset of a contractible open set U of \mathbb{R}^n . Then*

$$F_*(\partial J) = \partial K \iff F_*J = K.$$

Proof. This follows from Lemma 5.2.4 and since the boundary and pushforward operators are continuous and commute. \square

6 The part of a chain in an open set

Suppose $J \in \hat{\mathcal{B}}_n^1(W)$ and $U \subset W \subset \mathbb{R}^n$ is open and regular. We would like to define the *part of J in U* as an element $J|_U \in \hat{\mathcal{B}}_n^1(U)$ such that $\int_{J|_U} \omega = \int_J \omega$ for all $\omega \in \mathcal{B}_n^1(U)$. This is not always possible for open sets U that are not regular. For example, let U be the plane less the positive x -axis, and $J = ((1,0); 1)$, which is the unit simple 0-element at $(1,0)$.

Consider a function $f \in \mathcal{B}_0^1(U)$ satisfying $f(x, y) = \begin{cases} 1 & \text{for } \|x - 1\| \leq 1/2, y > 0 \\ 2 & \text{for } \|x - 1\| \leq 1/2, y < 0 \end{cases}$ If $((1, 0); e_1)|_U$

were well-defined in $\hat{\mathcal{B}}_1^1((U))$, then $\int_{((1,0);e_1)|_U} f dx = \lim_{y \rightarrow 0^+} \int_{((1,y);e_1)|_U} f dx = \lim_{y \rightarrow 0^-} \int_{((1,y);e_1)|_U} f dx$ and $y < 0$. However the first limit is one and the second is two (see [Har10] §6 for more details about chains defined in open sets.) However, we will be able to define $J|_U$ for open sets U whose boundaries miss a certain null set, i.e., a subset of \mathbb{R}^n with Lebesgue measure zero, related to J .

If $A = \sum(p_i; \alpha_i)$ is a Dirac k -chain in W and E is a subset of W , define the *part of A in E* by $A|_E := \sum(p_{i_j}; \alpha_{i_j})$ where $p_{i_j} \in E$.

For $0 \leq \ell \leq n$ and $t \in \mathbb{R}$, let $H_t^\ell = \{x \in W : x_\ell < t\}$ be the *open half-space* where $x = \sum_{i=1}^n x_i e_i$. Now fix ℓ and consider the family of $\{H_t^\ell\}_t$ of open half-spaces. If $\{A_i \in \mathcal{A}_n^0(W)\}_i$ is a Cauchy sequence in the B^1 norm, we shall show below (see Lemma 6.0.7) that the sequence $\{A_i|_{H_t^\ell} \in \mathcal{A}_n^0(W)\}_i$ is Cauchy in the B^1 norm for a.e. t . If $J = \lim_{i \rightarrow \infty} A_i$ in $\hat{\mathcal{B}}_n^1(W)$, we may then define

$$J|_{H_t^\ell} := \lim_{i \rightarrow \infty} A_i|_{H_t^\ell}$$

as a uniquely defined element of $\hat{\mathcal{B}}_n^1(W)$ a.e. t and $0 \leq \ell \leq n$. The next two lemmas 6.0.6 and 6.0.7 hold for both open and *closed half-spaces* in \mathbb{R}^n , the latter defined as $\{x \in \mathbb{R}^n : x_\ell \leq t\}$. Let Q_R be the open coordinate cube in \mathbb{R}^n with side length $2R$ and centered at the origin.

Lemma 6.0.6. *Suppose $D \in \mathcal{A}_n^0(Q_R)$ and $0 \leq \ell \leq n$. Then*

$$\int_{-R}^R \|D|_{H_t^\ell}\|_{B^1} dt \leq (4R + 1) \|D\|_{B^1}.$$

Proof. We first prove

$$\int_{-R}^R \|\Delta_u(q; \beta)|_{H_t^\ell}\|_{B^1} dt \leq (4R + 1) \|u\| \|\beta\| \quad (6.1)$$

for all $q \in Q_R$, $u \in \mathbb{R}^n$, and $\beta \in \Lambda_n(\mathbb{R}^n)$ such that $\Delta_u(q; \beta)$ is inside Q_R . Suppose u is in the direction of the ℓ -coordinate. Let y be the ℓ -coordinate of q and $y' = y + \|u\|$. Since $\Delta_u(q; \beta)$ is inside Q_R , then $-R \leq y \leq y' \leq R$. Thus the integral splits into three parts:

$$\begin{aligned} \int_{-R}^R \|\Delta_u(q; \beta)|_{H_t^\ell}\|_{B^1} dt &= \int_{-R}^y \|\Delta_u(q; \beta)|_{H_t^\ell}\|_{B^1} dt \\ &\quad + \int_y^{y'} \|\Delta_u(q; \beta)|_{H_t^\ell}\|_{B^1} dt \\ &\quad + \int_{y'}^R \|\Delta_u(q; \beta)|_{H_t^\ell}\|_{B^1} dt. \end{aligned}$$

The first integral is zero, the second is bounded by $\|u\|\|\beta\|$, and the third is bounded by $2R\|\Delta_u(\beta)\|_{B^1} \leq 2R\|u\|\|\beta\|$. This gives us an upper bound of $(2R+1)\|u\|\|\beta\|$ for the LHS of (6.1).

Next assume that u is orthogonal to the direction of the ℓ -coordinate. Then

$$\int_{-R}^R \|\Delta_u(q; \beta)|_{H_t^\ell}\|_{B^1} dt \leq 2R\|u\|\|\beta\|.$$

Inequality (6.1) follows from the triangle inequality. A similar, but easier, proof shows that $\int_{-R}^R \|(p; \alpha)|_{H_t^\ell}\|_{B^1} dt \leq 2R\|\alpha\|$ for all $q \in Q_R$ and $\alpha \in \Lambda_k(\mathbb{R}^n)$.

Now let $D \in \mathcal{A}_n^0(Q_R)$ and $\epsilon > 0$. There exists $D = \sum_{s=0}^j (p_s; \alpha_s) + \sum_{i=1}^m \Delta_{u_i}(q_i; \beta_i)$ where each $\Delta_{u_i}(q_i; \beta_i)$ and $(p_s; \alpha_s)$ is inside Q_R and $\|D\|_{B^1} > \sum_{s=0}^j \|\alpha_s\| + \sum_{i=1}^m \|u_i\|\|\beta_i\| - \epsilon$.

Therefore,

$$\begin{aligned} \int_{-R}^R \|D|_{H_t^\ell}\|_{B^1} dt &\leq \sum_{s=0}^j \int_{-R}^R \|(p_s; \alpha_s)|_{H_t^\ell}\|_{B^1} dt + \sum_{i=1}^m \int_{-R}^R \|\Delta_{u_i}(q_i; \beta_i)|_{H_t^\ell}\|_{B^1} dt \\ &\leq (4R+1) \left(\sum_{s=0}^j \|\alpha_s\| + \sum_{i=1}^m \|u_i\|\|\beta_i\| \right) \\ &< (4R+1)(\|D\|_{B^1} + \epsilon). \end{aligned}$$

The result follows since this inequality holds for all $\epsilon > 0$.

□

Lemma 6.0.7. *If $\{D_i\}$ is a sequence in $\mathcal{A}_n^0(Q_R)$ such that $\sum_{i=1}^\infty \|D_i\|_{B^1} < \infty$, then $\sum_{i=1}^\infty \|D_i|_{H_t^\ell}\|_{B^1} < \infty$ for almost all $t \in \mathbb{R}$ and $0 \leq \ell \leq n$.*

Proof. Since each $\|D_i|_{H_t^\ell}\|_{B^1}$ is piecewise linear as a function of t , and thus measurable, we may apply Fatou's Lemma and Lemma 6.0.6 to deduce

$$\int_{-R}^R \left(\sum_{i=1}^\infty \|D_i|_{H_t^\ell}\|_{B^1} \right) dt \leq \sum_{i=1}^\infty \int_{-R}^R \|D_i|_{H_t^\ell}\|_{B^1} dt \leq \sum_{i=1}^\infty (4R+1)\|D_i\|_{B^1} < \infty.$$

Thus $\sum_{i=1}^\infty \|D_i|_{H_t^\ell}\|_{B^1} < \infty$ a.e. t .

□

We say an n -cell Q is *partly open and closed* if it contains some of its $(n-1)$ -faces, but not all. Let \mathring{Q} be the interior of Q and \overline{Q} its closure. In this case it follows that if $\omega \in \mathcal{B}_k^r(\mathring{Q})$, then ω extends to an element $\mathcal{B}_k^r(W)$ where W is a neighborhood of Q . If Z is a union of hyperplanes of \mathbb{R}^n , we

say that Q is Z -compatible if the faces of Q are not contained in Z . In the next result we assume that Q is a coordinate n -cube for a fixed orthonormal basis \mathbb{R}^n , because this is all we need and it simplifies the exposition.

Theorem 6.0.8. *Let $J \in \hat{\mathcal{B}}_n^1(U)$. There exists a null set $Z_J \subset \mathbb{R}^n$ that is a union of hyperplanes such that if $Q \subset U$ is a partly open and closed n -cube and (Z_J) -compatible, then $J|_{Q \in \hat{\mathcal{B}}_n^1(Q)}$ is well-defined with $\int_{J|_Q} \omega = \int_J \omega$ for all $\omega \in \hat{\mathcal{B}}_n^1(Q)$. Furthermore, if $J_i \rightarrow J$ with $J_i \in \hat{\mathcal{B}}_n^1(U)$, then $J|_Q = \lim_{i \rightarrow \infty} J_i|_Q$ for all Q which are $(Z_J \cup_i Z_{J_i})$ -compatible.*

Proof. Choose $D_j \rightarrow J$ in $\hat{\mathcal{B}}_n^1(U)$ with $D_j \in \mathcal{A}_n^0(U)$ and $\|D_j - D_{j+1}\|_{B^1} < 2^{-j}$. Therefore, $\sum_{j=1}^{\infty} \|D_j - D_{j+1}\|_{B^1} < \infty$. By Lemma 6.0.7 $\sum_{j=1}^{\infty} \|(D_j - D_{j+1})|_{H_t^\ell}\|_{B^1} < \infty$ for almost every t . For such t , $D_j|_{H_t^\ell}$ forms a Cauchy sequence in $\hat{\mathcal{B}}_n^1(U \cap H_t^\ell)$ and we may define

$$J|_{H_t^\ell} := \lim_{j \rightarrow \infty} D_j|_{H_t^\ell}.$$

If $\{D'_i\}$ is another sequence which has the above properties and tends to J , then by taking subsequences, we may assume that $\sum \|D_i - D'_i\|_{B^1} < \infty$. If H_t^ℓ is $(Z_J \cup_i Z_{D_i})$ -compatible, then $D_i|_{H_t^\ell}$ and $D'_i|_{H_t^\ell}$ tend to the same limit as $i \rightarrow \infty$. The integral condition holds, for if $\omega \in \mathcal{B}_n^1(H_t^\ell)$, then $\int_{D_i|_{H_t^\ell}} \omega = \int_{D_i} \omega$ for hyperplanes H_t^ℓ compatible with $\cup_i Z_{D_i}$. Therefore, $\int_{J|_{H_t^\ell}} \omega = \lim_{i \rightarrow \infty} \int_{D_i|_{H_t^\ell}} \omega = \lim_{i \rightarrow \infty} \int_{D_i} \omega = \int_J \omega$.

This method holds for half-planes taken in any direction, so we may apply it to $J|_{H_t^\ell}$ using a different face of Q . A simple inductive argument establishes the result. The half-planes are either open or closed, and thus Q can be partly open and closed. \square

Theorem 6.0.9. *If $J_i \rightarrow J$ in $\hat{\mathcal{B}}_n^1(U)$, then $J_i|_Q \rightarrow J|_Q$ in $\hat{\mathcal{B}}_n^1(U)$ where $Q \subset U$ is an n -cube which is $(Z_J \cup_i Z_{J_i})$ -compatible.*

Proof. Suppose $J_i \rightarrow J$ in $\hat{\mathcal{B}}_n^1(U)$. Let $\omega \in \mathcal{B}_n^1(U)$. Then $\omega|_Q \in \mathcal{B}_n^1(Q)$ and by Theorem 6.0.8 $\int_{J_i|_Q} \omega = \int_{J_i} \omega \rightarrow \int_J \omega = \int_{J|_Q} \omega$. \square

This result easily extends to any finite collection of n -cubes $\{Q_i\}$ that are (Z_J) -compatible. We may assume each Q_i is partly open and closed so that the collection is non-overlapping and the union $W = \cup Q_i$ is open. In this way, we may use Theorem 6.0.8 to define $J|_W$ satisfying $\int_{J|_W} \omega = \int_J \omega$ for all ω supported in W .

7 Existence of area minimizers for surfaces spanning a Jordan curve in \mathbb{R}^3

7.1 A complete space of spanning dipole surfaces

We shift our attention from the chain complex of differential chains $\hat{\mathcal{B}}_k^{s+1}$ to the smaller chainlet complex $\mathcal{Ch}_k^s \subset \hat{\mathcal{B}}_k^{s+1}$, where compactness will be easier to establish. For the rest of the paper, we will assume $n = 3$ for simplicity. However, all results hold for codimension one surfaces in \mathbb{R}^n for $n \geq 2$. The extension to codimension j is obtained simply by replacing the vector fields X and Y with j -vector fields. Figure 5 illustrates a codimension two dipole surface in \mathbb{R}^3 .

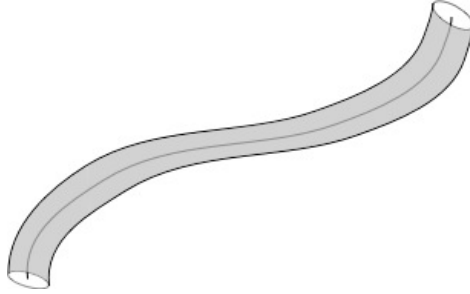


Figure 5: A codimension two dipole surface

Fix $R > 0$ and let $\Omega_R \subset \mathbb{R}^n$ be the open ball about the origin of radius R . For convenience, recall

$$\mathcal{I}_2^1(\Omega_R) \subset \overline{\mathcal{I}_2^1(\Omega_R)} \subset \mathcal{Ch}_2^1(\Omega_R) \subset \hat{\mathcal{B}}_2^2(\Omega_R).$$

Let γ be a smooth Jordan curve in $\Omega_R - \{0\}$ and $\tilde{\gamma} \in \mathcal{Ch}_1^0(\Omega_R)$ the 1-chain representing γ (see Theorem 2.4.3). Let $L\gamma$ denote the union of lines connecting points in γ to the origin. We may suppose that $L\gamma$ is transverse to γ so that $L\gamma$ is a finite union of smoothly embedded surfaces. (If necessary, shift the origin $\{0\}$ slightly. Transversality will hold a.e. point in a neighborhood of $\{0\}$ by Sard's Theorem. We will later show that our solution to Plateau's problem is independent of the choice of the cone point.)

Let $Y \in \mathcal{V}^1(\Omega_R)$ be a unit Lipschitz vector field defined on Ω_R that is orthogonal to $L\gamma$ at each $p \in \gamma$. We can choose Y so that $Y(p) \wedge K(p) \wedge u(p)$ is positively oriented where $K(p)$ is the unit vector at p in the direction of the line segment Lp , and $u(p)$ is the unit vector tangent to γ at p . We are only interested in the restriction of Y to γ (where it is assumed to be transverse), and the fact that Y extends to a Lipschitz vector field in a neighborhood of γ . It follows that if Y_1 and Y_2 are two such vector fields, then $P_{Y_1}\tilde{\gamma} = P_{Y_2}\tilde{\gamma}$. Our solution to Plateau's problem will nominally depend on Y , but we will show this is artificial.

7.1.1 Span

We need a general way to define the integral monopole chain \bar{S} , as a function of the integral dipole 2-chain S , using our continuous operators so that we can take limits. The next two Lemmas provide this:

Lemma 7.1.1. *If $S = \sum_i P_{X_i} \tilde{\tau}_i \in \mathcal{I}_2^1(\Omega_R)$, then $\|\bar{S}\|_{B^1} \leq \|S\|_{B^2}$ where $\bar{S} = \sum_j E_{X_j} \tilde{\tau}_j$.*

Proof. If τ is a 2-cell smoothly embedded in Ω_R , then τ is approximated by Dirac chains of the form $\sum_s (p_s; \alpha_s)$ in the B^1 norm where α_s are certain tangent 2-vectors of τ at $p_s \in \tau$ (see the proof of Theorem 2.10.2 in [Har10]). Since all simple dipole 2-elements $(p_s; X(p_s) \otimes \alpha_s)$ have positive orientation, so do the simple 3-elements $(p_s; X(p_s) \wedge \alpha_s)$. Then $E_X \tilde{\tau}$ is approximated in the B^1 norm by Dirac 3-chains of the form $\sum_s (p_s; X(p_s) \wedge \alpha_s)$. It also follows that the integral dipole cell $P_X \tilde{\tau}$ is approximated in the B^2 norm by Dirac 2-chains of dipole order 1 of the form $\sum_s (p_s; X(p_s) \otimes \alpha_s)$.

Suppose now that $S = \sum_i P_{X_i} \tilde{\tau}_i$. Then for each $\epsilon > 0$, there exists $\sum_i \sum_j (p_{ij}; X_i(p_{ij}) \otimes \alpha_{ij}) \in \mathcal{A}_2^1(\Omega_R)$ with $\|S - \sum_i \sum_j (p_{ij}; X_i(p_{ij}) \otimes \alpha_{ij})\|_{B^2} < \epsilon$. It follows that $\|\bar{S} - \sum_i \sum_j (p_{ij}; X_i(p_{ij}) \wedge \alpha_{ij})\|_{B^1} < \epsilon$. Hence $\|\bar{S}\|_{B^1} \leq \|\sum_i \sum_j (p_{ij}; X_i(p_{ij}) \wedge \alpha_{ij})\|_{B^1} + \epsilon = \|\sum_i \sum_j (p_{ij}; X_i(p_{ij}) \otimes \alpha_{ij})\|_{B^2} + \epsilon < \|S\|_{B^2} + 2\epsilon$. \square

Lemma 7.1.2. *If $S = \sum_{i=1}^s P_{X_i} \tilde{\tau}_i$ is an integral dipole chain with $\partial S = P_Y \tilde{\gamma}$, then $\bar{S} = \kappa(S - E_Y \tilde{\gamma})$ where $\bar{S} = \sum_{i=1}^s E_{X_i} \tilde{\tau}_i$.*

Proof. Since $\sum_{i=1}^s P_{X_i} \partial \tilde{\tau}_i = P_Y \tilde{\gamma}$, it follows that $\sum_{i=1}^s E_{X_i} \partial \tilde{\tau}_i = E_Y \tilde{\gamma}$. By Theorem 5.2.2 $\kappa \partial + \partial \kappa = I$, and therefore

$$\partial \sum_{i=1}^s E_{X_i} \tilde{\tau}_i = \sum_{i=1}^s P_{X_i} \tilde{\tau}_i - E_{X_i} \partial \tilde{\tau}_i = \sum_{i=1}^s P_{X_i} \tilde{\tau}_i - E_Y \tilde{\gamma} = \partial \kappa \left(\sum_{i=1}^s P_{X_i} \tilde{\tau}_i - E_Y \tilde{\gamma} \right)$$

According to Corollary 5.2.4 we conclude that $\sum_{i=1}^s E_{X_i} \tilde{\tau}_i = \kappa(\sum_{i=1}^s P_{X_i} \tilde{\tau}_i - E_Y \tilde{\gamma})$. \square

Suppose $S = \sum_{i=1}^s P_{X_i} \tilde{\tau}_i \in \mathcal{I}_2^1(\Omega_R)$ is an integral dipole 2-chain. Let $\bar{S} := \sum_{i=1}^s E_{X_i} \tilde{\tau}_i \in \mathcal{I}_3^0(\Omega_R) \subset \mathcal{Ch}_3^0(\Omega_R)$. Then \bar{S} is an integral monopole 3-chain which is the “infinitesimal fill” of the integral dipole surface S . Let $Z_{\bar{S}}$ be a null set of hyperplanes satisfying the conditions of Theorem 6.0.8. Let $\gamma' \subset \Omega_R$ be a closed loop linking γ once. For each $0 < \delta < d(\gamma', \gamma)$, where d is the Hausdorff metric, let $N_\delta(\gamma')$ be a neighborhood of γ' consisting of a finite union of non-overlapping 3-cubes meeting γ' with edge length δ and which are $Z_{\bar{S}}$ -compatible. Now suppose $p \in \text{supp}(\bar{S})$ and $p \notin \gamma$. For each $0 < \delta < d(p, \gamma)$, let $Q_\delta(p)$ be a cubical neighborhood of p with edge length δ and which

is $Z_{\overline{S}}$ -compatible. It follows from Theorem 6.0.8 that $\overline{S}|_{N_\delta(\gamma')}$ and $\overline{S}|_{Q_\delta(p)}$ are well-defined elements of $\hat{\mathcal{B}}_3^1(\Omega_R)$, and we may therefore integrate Lipschitz 3-forms over them.

Proposition 7.1.3. *If S is an integral dipole 2-chain with $\partial S = P_Y \tilde{\gamma}$, then*

$$A(S) = \int_{\kappa(S - E_Y \tilde{\gamma})} dV.$$

Proof. This follows from Lemma 7.1.2. □

For $S \in \overline{\mathcal{I}_2^1(\Omega_R)}$, define $A(S) := \int_{\kappa(S - E_Y \tilde{\gamma})} dV$. It follows that the area functional A is continuous: Suppose $S_i \rightarrow S$ in the B^2 norm where $S_i \in \mathcal{I}_2^1(\Omega_R)$. Then $\kappa(S_i - E_Y \tilde{\gamma}) \rightarrow \kappa(S - E_Y \tilde{\gamma})$ in $\overline{\mathcal{I}_3^0(\Omega_R)}$ by Lemma 7.1.1 and hence $A(S_i) \rightarrow A(S)$. Let \mathcal{H}^k denote k -dimensional Hausdorff measure.

Corollary 7.1.4. *If $S = \sum_{i=1}^s P_{X_i} \tilde{\tau}_i \in \mathcal{I}_2^1(\Omega_R)$ is an integral dipole chain, and the cells $\{\tau_i\}$ are non-overlapping, then $A(S) = \mathcal{H}^2(\cup \tau_i)$.*

Proof. Using the integral relation (3.1), Theorem 2.4.3, the assumption that the component of X_i orthogonal to τ_i is unit, the definition of Hausdorff measure for smoothly embedded cells, and additivity of Hausdorff measure for non-overlapping sets with smooth boundaries, we have

$$A(S) = \sum_{i=1}^s A(P_{X_i} \tilde{\tau}_i) = \sum_{i=1}^s \int_{E_{X_i} \tilde{\tau}_i} dV = \sum_{i=1}^s \int_{\tilde{\tau}_i} i_{X_i} dV = \sum_{i=1}^s \int_{\tau_i} i_{X_i} dV = \sum_{i=1}^s \mathcal{H}^2(\tau_i) = \mathcal{H}^2(\cup_i \tau_i).$$

(The last integral is the Riemann integral.) □

Definition 7.1.5. *We say that $S \in \overline{\mathcal{I}_2^1(\Omega_R)}$ spans γ (with respect to Y) if*

- $\partial S = P_Y \tilde{\gamma}$;
- If γ' is a simple closed curve linking γ with linking number one, then a.e. $0 < \delta < d(\gamma', \gamma)$ the differential chain $\overline{S}|_{N_\delta(\gamma')}$ is well-defined in $\hat{\mathcal{B}}_3^1(\Omega_R)$ and $\int_{\overline{S}|_{N_\delta(\gamma')}} dV \geq \delta^2$.
- If $p \in \text{supp}(\overline{S}) - \gamma$, then a.e. $0 < \delta < d(p, \gamma)$ the differential chain $\overline{S}|_{Q_\delta}$ is well-defined in $\hat{\mathcal{B}}_3^1(\Omega_R)$ and $\int_{\overline{S}|_{Q_\delta}} dV \geq \delta^2$.

Proposition 7.1.6. *Span is well-defined.*

Proof. The first property is valid since ∂ is continuous.

For the other two properties, we use Theorem 6.0.8 which says the part of a chain in an open set is well-defined and continuous for the spaces $\mathcal{B}_k^1(\Omega_R)$:

Suppose γ' links γ once and the cubes of $N_\delta(\gamma')$ are $(Z_{\bar{S}} \cup_i Z_{\bar{S}_i})$ -compatible. Then $\bar{S}|_{N_\delta}$ and $\bar{S}_i|_{N_\delta}$ are well-defined with $\bar{S}_i|_{N_\delta} \rightarrow \bar{S}|_{N_\delta}$ by Theorem 6.0.8. Therefore $\int_{\bar{S}|_{N_\delta}} dV = \lim_{i \rightarrow \infty} \int_{\bar{S}_i|_{N_\delta}} dV \geq \delta^2$.

Let $p \in \text{supp}(\bar{S}) - \gamma$. We show there exists $p_{i_j} \in \text{supp}(\bar{S}_{i_j}) - \gamma$ with $\|p_{i_j} - p\| \rightarrow 0$. Suppose not. There exists $\epsilon_0 < d(p, \gamma)$ such that if $0 < \epsilon < \epsilon_0$, then there exists N so that $\text{supp}(\bar{S}_{i_j}) \cap Q_\epsilon(p) = \emptyset$ for all $j > N$. We know that $Q_\epsilon(p)$ and $Q_{\epsilon/2}(p)$ are $(\bar{S} \cup_i \bar{S}_i)$ -compatible a.e. $\epsilon > 0$. Also $p \in \text{supp}(\bar{S}|_{Q_{\epsilon/2}(p)})$ implies that $\bar{S}|_{Q_{\epsilon/2}(p)} \neq 0$ since the support of a nonzero chain is nonempty (2.3.3). Since $\bar{S}|_{Q_{\epsilon/2}(p)} \in \mathcal{B}_3^1(Q_{\epsilon/2}(p))$ there exists $\omega_1 \in \mathcal{B}_3^1(Q_{\epsilon/2}(p))$ with $\int_{\bar{S}|_{Q_{\epsilon/2}(p)}} \omega_1 > 0$. We can extend ω_1 to an element of $\mathcal{B}_3^1(Q_\epsilon(p))$ which vanishes at the faces of $Q_\epsilon(p)$ since $Q_{\epsilon/2}(p)$ is a regular open set. Then $\int_{\bar{S}_{i_j}|_{Q_\epsilon(p)}} \omega_1 = 0$ since $\text{supp}(\bar{S}_{i_j}) \cap Q_\epsilon(p) = \emptyset$. Since \bar{S} is positively oriented, $\int_{\bar{S}|_{Q_\epsilon(p)}} \omega_1 \geq \int_{\bar{S}|_{Q_{\epsilon/2}(p)}} \omega_1 > 0$. Since $\bar{S}|_{Q_\epsilon(p)} = \lim_{j \rightarrow \infty} \bar{S}_{i_j}|_{Q_\epsilon(p)}$ we know $\int_{i_j|_{Q_\epsilon(p)}} \omega_1 = \lim_{j \rightarrow \infty} \int_{\bar{S}_{i_j}|_{Q_\epsilon(p)}} \omega_1 = 0$, a contradiction. Therefore, there exists $p_{i_j} \in \text{supp}(\bar{S}_{i_j}) - \gamma$ with $\|p_{i_j} - p\| \rightarrow 0$. Since $p_{i_j} \rightarrow p$, there exists $\epsilon_{i_j} \rightarrow \epsilon$ with $Q_{\epsilon_{i_j}}(p_{i_j}) \subset Q_\epsilon(p)$, and thus $\int_{\bar{S}_{i_j}|_{Q_\epsilon(p)}} dV \geq \int_{\bar{S}_{i_j}|_{Q_{\epsilon_{i_j}}(p_{i_j})}} dV > \epsilon_{i_j}^2$. Therefore $\int_{\bar{S}|_{Q_\epsilon}} dV = \lim_{j \rightarrow \infty} \int_{\bar{S}_{i_j}|_{Q_\epsilon(p)}} dV \geq \epsilon^2$, and we deduce that S spans γ . \square

The first condition assures us that the support of the boundary of S is γ since $\text{supp}(P_Y \tilde{\gamma}) = \text{supp}(\tilde{\gamma}) = \gamma$ (see Propositions 5.2.3 and 5.2.4 of [Har10]), the second that there are “no holes” in S as in Figure 6, and the last condition guarantees that the surface S is not “too thin”.

Proposition 7.1.7. *If S is an integral dipole 2-chain, then $\text{supp}(S) = \text{supp}(\bar{S}) = \text{supp}(E_X S) = \text{supp}(P_X S)$.*

Proof. This follows since $\text{supp}(E_X \tilde{\tau}) = \text{supp}(P_X \tilde{\tau}) = \text{supp}(\tilde{\tau}) = \tau$ and there is no cancellation when forming finite sums since all dipole cells are positively oriented. \square

Set $c = R\mathcal{H}^1(\gamma)$.

Definition 7.1.8. *Let $\mathcal{S}_2(\Omega_R, \gamma, Y) := \{S \in \overline{\mathcal{I}_2^1(\Omega_R)} : A(S) \leq c, S \text{ spans } \gamma, \text{ and } \text{supp}(S) \subset \Omega_R\}$.*

Our *candidate surfaces* for Plateau’s problem in n -space are the supports of elements $S \in \mathcal{S}_2(\Omega_R, \gamma, Y)$. We now have established definitions of *surface*, *area* and *span* for which we can solve problems of the calculus of variations such as Plateau’s problem.

Proposition 7.1.9. *Suppose γ is a smooth Jordan curve embedded in $\Omega_R - \{0\}$. Then $\mathcal{S}_2(\Omega_R, \gamma, Y)$ is a complete subspace of $\overline{\mathcal{I}_2^1(\Omega_R)}$.*

Proof. Suppose $S_i \in \mathcal{S}_2(\Omega_R, \gamma, Y) \cap \mathcal{I}_2^1(\Omega_R)$ and $S_i \rightarrow S$ in $\overline{\mathcal{I}_2^1(\Omega_R)}$. By the remark after Proposition 7.1.3 $A(S_i) \rightarrow A(S)$, and hence $A(S) \leq c$. We know that S spans γ by Proposition 7.1.6

We prove that $\text{supp}(\overline{S}) \subset \Omega_R$: Suppose there exists $p \in \text{supp}(\overline{S})$ and $p \notin \Omega_R$. Then there exists $Q_\epsilon(p)$ missing Ω_R and a form ω supported in $Q_\epsilon(p)$ with $f_{\overline{S}}\omega \neq 0$ (See [Har10] Theorem 5.2.2.). Since $S_i \rightarrow S$, we know that $f_{\overline{S}_i}\omega \neq 0$ for sufficiently large i . But this contradicts the assumption that $\text{supp}(\overline{S}_i) \subset \Omega_R$. □

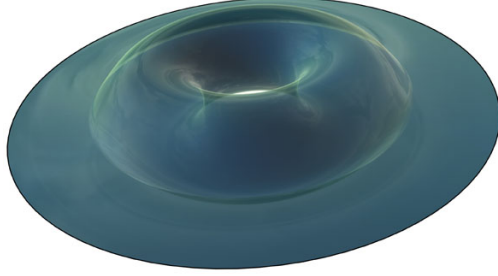


Figure 6: A non-spanning surface of the unit circle (drawing by H. Pugh)

7.1.2 Existence of an integral dipole surface that spans γ

Proposition 7.1.10. *Suppose γ is a smooth Jordan curve embedded in $\Omega_R - \{0\}$. Then $\mathcal{S}_2(\Omega_R, \gamma, Y)$ is nonempty.*

Proof. Recall the set $L\gamma$ is a finite union of smoothly embedded surfaces whose topological boundary is γ . Smooth out a neighborhood of the cone point to obtain a surface C with boundary γ which is also a union of finitely many piecewise smooth embedded surfaces. Let \tilde{C} be its representative in $\mathcal{Ch}_2^0(\Omega_R)$. Then construct a smooth unit vector field orthogonal to C extending Y . It follows that $P_Y\tilde{C} \in \mathcal{Ch}_2^1(\Omega_R)$ is an integral dipole chain spanning γ . □

7.2 A lower bound on area

Proposition 7.2.1. *Suppose γ is a smooth Jordan curve embedded in $\Omega_R - \{0\}$. There exists a constant $a_0 > 0$ such that if $S \in \mathcal{S}_2(\gamma, \Omega_R, Y)$, then $A(S) \geq a_0$.*

Proof. Since $\text{supp}(S)$ is closed in Ω_R , there exists $R' < R$ with $\text{supp}(S) \subset \Omega_{R'}$. By Sard's theorem, we may project γ onto a plane P so that the projection $\pi\gamma$ is 1-1 except at finitely many points. The set $P - \pi\gamma$ has finitely many connected components. Let U be the unbounded component of $P - \pi\gamma$. We show that $A(S)$ exceeds the total area of the components of $P - \pi\gamma$ whose closures meet the closure of U along an arc of $\pi\gamma$. Let Y be such a component. Then its area a_0 is nonzero. For a square $Q \subset Y$, let $\hat{Q} = \pi^{-1}Q$. Choose $p_0 \in Y$.

For each $\epsilon > 0$, there exists $\delta > 0$ and a non-overlapping collection of squares $\{Q_i \subset Y\}_{i=1}^s$ with side length $\delta \leq d(p_0, \pi\gamma)$ such that $Y - \cup_{i=1}^s Q_i$ has area $< \epsilon$, $p_0 \in Q_0$, and the \hat{Q}_i are $(Z_{\bar{S}})$ -compatible. Now the interval $\pi^{-1}(p_0) \cap \Omega_{R'}(0)$ can be augmented to form a smooth loop $\ell_0 \subset \Omega_R$ linking γ once. Let T_0 be a δ -neighborhood of ℓ_0 consisting of the union of \hat{Q}_0 with a non-overlapping union of 3-cubes with diameter δ meeting ℓ_0 which are disjoint from $\text{supp}(S)$. Since S spans γ , we know $f_{\bar{S}|_{\hat{Q}_0}} = f_{\bar{S}|_{T_0}} \geq \delta^2$. Therefore,

$$A(S) = \int_{\bar{S}} dV \geq \sum_{i=1}^s \int_{\bar{S}|_{\hat{Q}_i}} dV \geq \sum_{i=1}^s \delta^2 > a_0 - \epsilon > 0.$$

Thus $A(S) \geq a_0$. □

7.3 Compactness

Proposition 7.3.1. *Suppose γ is a smooth Jordan curve embedded in $\Omega_R - \{0\}$. Then $\mathcal{S}_2(\Omega_R, \gamma, Y)$ is totally bounded in $\mathcal{Ch}_2^1(\Omega_R)$.*

Proof. First observe that if $P_X \tau \in \mathcal{Ch}_2^1(\Omega_R)$, then it is approximated by finite sums $\sum_{i=1}^s P_{X(q_i)}(q_i; \beta_i)$ in the B^1 norm as in Lemma 7.1.1.

For $k \in \mathbb{Z}$, $k \geq 1$, let $\mathcal{Q}(k)$ be all rationals $j/2^k$ with $j \in \mathbb{Z}$ and $0 \leq |j| \leq 2^k$. Let $\mathcal{Z}(k)$ be the subset of integral dipole 2-chains $\sum_{i=1}^s P_{v_i}(q_i; \beta_i) \in \mathcal{Ch}_2^1(\Omega_R)$ and such that

- q_i is a vertex of the binary lattice with edge length 2^{-k} and subdividing Ω_R ;
- $v_i/k \in \mathbb{R}^3$ has coordinates in $\mathcal{Q}(k)$ so that $\|v_i\| \leq k$;

- β_i is a 2-vector with coordinates in $\mathcal{Q}(k)$ written in terms of an orthonormal basis of $\Lambda_2(\mathbb{R}^3)$;
- $\sum_{i=1}^s M(\beta_i) \leq c$.

It follows that $\mathcal{Z}(k)$ contains only finitely many chains.

Let $S \in \mathcal{S}_2(\Omega_R, \gamma, Y)$. Then S spans γ , $A(S) \leq c$, and $\text{supp}(S) \subset \Omega_R$. It follows that S can be approximated by an affine dipole 2-chain S' , which can itself be approximated by a Dirac dipole 2-chain $A = \sum_{i=1}^s P_{u_i}(p_i; \alpha_i) \in \mathcal{A}_2^1(\Omega_R)$ with $\|u_i\| \leq k$, $\sum_{i=1}^s \|\alpha_i\| \leq c$. All approximations are in $\mathcal{C}h_2^1(\Omega_R)$.

Now A can be approximated by an element of $\mathcal{Z}(k)$ as follows: For each dipole 2-vector $P_{u_i}(p_i; \alpha_i)$ we know p_i lies in some cube Q' of the binary lattice subdividing Ω_R . Let p'_i be a vertex of Q' . Let u'_i have coordinates in $\mathcal{Q}(k)$ s.t. $\|u_i - u'_i\| \leq 2^{-k}$. Let α'_i be a 2-vector with coordinates in $\mathcal{Q}(k)$ s.t. $\|\alpha_i - \alpha'_i\| < 2^{-k}\|\alpha_i\|$. Then $A' = \sum_{i=1}^s P_{u'_i}(p'_i; \alpha'_i) \in \mathcal{Z}(k)$. It follows that

$$\begin{aligned}
\|A - A'\|_{B^1} &\leq \|\sum_i P_{u_i}(p_i; \alpha_i) - P_{u'_i}(p_i; \alpha_i)\|_{B^1} + \|\sum_i P_{u'_i}(p_i; \alpha_i) - P_{u'_i}(p'_i; \alpha'_i)\|_{B^1} \\
&\quad + \|\sum_i P_{u'_i}(p'_i; \alpha'_i) - P_{u'_i}(p'_i; \alpha'_i)\|_{B^1} \\
&\leq \sum_{i=1}^s \|P_{u_i - u'_i}(p_i; \alpha_i)\|_{B^1} + \sum_{i=1}^s \|P_{u'_i}((p_i; \alpha_i) - (p'_i; \alpha'_i))\|_{B^1} + \sum_{i=1}^s \|P_{u'_i}(p'_i; \alpha_i - \alpha'_i)\|_{B^1} \\
&\leq \sum_{i=1}^s \|u_i - u'_i\| \|\alpha_i\| + \|u'_i\| \|p_i - p'_i\| \|\alpha_i\| + \|u'_i\| \|\alpha_i - \alpha'_i\| \\
&\leq k \sum_{i=1}^s 2^{-k} \|\alpha_i\| + (1 + 2^{-k}) 2^{-k} \|\alpha_i\| + (1 + 2^{-k}) 2^{-k} \|\alpha_i\| \\
&\leq k 2^{-k} \sum_{i=1}^s \|\alpha_i\| + 2(1 + 2^{-k}) \|\alpha_i\| \\
&\leq 4k 2^{-k} \sum_{i=1}^s \|\alpha_i\| \\
&\leq 4kc 2^{-k}.
\end{aligned}$$

This proves that $\mathcal{S}_2(\Omega_R, \gamma, Y)$ is totally bounded. □

Theorem 7.3.2. *Suppose γ is a smooth Jordan curve embedded in $\Omega_R - \{0\}$. Then $\mathcal{S}_2(\Omega_R, \gamma, Y)$ is compact.*

Proof. This follows from Propositions 7.1.9 and 7.3.1. □

Theorem 7.3.3. *Suppose γ is a smooth Jordan curve embedded in $\Omega_R - \{0\}$. There exists $S_0 \in \mathcal{S}_2(\Omega_R, \gamma, Y)$ spanning γ with minimal area $A(S_0)$.*

Proof. Let $m = \inf\{A(S) : S \in \mathcal{S}_2(\Omega_R, \gamma, Y)\}$. By 7.2.1 we know $m > 0$. There exists a sequence $\{S_i\} \in \mathcal{S}_2(\Omega_R, \gamma, Y)$ such that $A(S_i) \rightarrow m$ as $i \rightarrow \infty$. By compactness of $\mathcal{S}_2(\Omega_R, \gamma, Y)$ (7.3.2) and continuity of A , there exists a subsequential limit $S_0 \in \mathcal{S}_2(\Omega_R, \gamma, Y)$ with $A(S_0) = m$. \square

Show that this is independent of choice of R, Y , and $\{0\}$.

It is clearly impossible to find spanning chains with smaller area using chains not supported in Ω_R , but we have to prove this. The technical difficulty is that pushforward of a dipole chain can change not only area but also “dipole distance” between layers since $F_*(p; u \otimes \alpha) = (F(p); F_*u \otimes F_*\alpha)$. We have to “renormalize” a dipole surface after pushforward, so that it becomes an integral dipole surface.

Suppose $P_X \tilde{\tau}$ is an integral dipole 2-cell where τ is contained in the ball Ω_{R_1} of radius $R_1 > R$ about the origin. There exists $0 < R_2 < R$ such that $\gamma \subset \Omega_{R_2}$. Let $F : \Omega_{R_1} \rightarrow \Omega_R$ be a diffeomorphism that is the identity on Ω_{R_2} . By Theorem 8.5.1 of [Har10] $F_*(P_X \tilde{\tau}) = P_{FX}(F_* \tilde{\tau})$. There exists a unique vector field V defined on $F \tilde{\tau}$ so that $P_V F_* \tilde{\tau}$ is integral and V is in the same direction as $F_* X$. Let $f : \tau \rightarrow \mathbb{R}$ be defined by $f(p) = \|V(p)\| / \|F_* X(p)\|$. Then $m_f F_* P_X \tilde{\tau} = m_f P_{FX} F_* \tilde{\tau} = P_V F_* \tilde{\tau}$. It follows that f is as smooth as F, F^{-1} , and τ . Use the Whitney Extension Theorem to extend f to Ω_{R_1} . According to Theorem 4.1.4 of [Har10] the operator “multiplication by a function” $m_f : \mathcal{C}h_2^1(\Omega_{R_1}) \rightarrow \mathcal{C}h_2^1(\Omega_{R_1})$ is continuous with $m_f(p; u \otimes \alpha) = f(p)(p; u \otimes \alpha)$.

Suppose there exists $S' \in \mathcal{S}_2(\Omega_{R_1}, \gamma, Y)$ which has smaller area than S_0 . Then S' is approximated by integral dipole chains of the form $\sum_{i=1}^s P_{X_i} \tilde{\tau}_i$ where $\tau_i \subset \Omega_{R_1}$ with $\partial \sum_{i=1}^s P_{X_i} \tilde{\tau}_i = P_Y \tilde{\gamma}$. Let $S'' = \sum_{i=1}^s P_{V_i} F_* \tilde{\tau}_i$. Since $F = I$ on Ω_{R_2} , we have

$$\begin{aligned} \partial S'' &= \sum_{i=1}^s \partial P_{V_i} F_* \tilde{\tau}_i = \sum_{i=1}^s \partial m_{f_i} F_* P_{X_i} \tilde{\tau}_i \\ &= \sum_{i=1}^s (m_{f_i} \partial + m_{f'_i}) F_* P_{X_i} \tilde{\tau}_i \\ &= P_Y \tilde{\gamma}. \end{aligned}$$

The other conditions hold for S'' to span γ in Ω_R since they are preserved under diffeomorphisms and the area functional is continuous. Therefore $S'' \in \mathcal{S}_2(\Omega_R, \gamma, Y)$. This contradicts the minimality of the area of $S_0 \in \mathcal{S}_2(\Omega_R, \gamma, Y)$. We next show that S_0 is independent of the choice of the vector field Y . Suppose Y_1 and $Y_2 \in \mathcal{V}^1(\Omega_R)$ are unit Lipschitz vector fields defined on Ω_R that are orthogonal

to γ at each $p \in \gamma$. Then

$$(P_{Y_1} - P_{Y_2})\tilde{\gamma} = P_{Y_1 - Y_2}\tilde{\gamma} = \partial E_{Y_1 - Y_2}\tilde{\gamma}.$$

Now adding $E_{Y_1 - Y_2}\tilde{\gamma}$ to S_0 does not change anything important. In particular, $\partial(S_0 + E_{Y_1 - Y_2}\tilde{\gamma}) = P_{Y_1}\tilde{\gamma}$ and $A_{Y_1}(S_0 + E_{Y_1 - Y_2}\tilde{\gamma}) = \int_{S_0 + E_{Y_1 - Y_2}\tilde{\gamma} - E_{Y_1}\tilde{\gamma}} dV = \int_{S_0 - E_{Y_2}\tilde{\gamma}} dV = A_{Y_2}(S_0)$. It is not hard to show that $S_0 + E_{Y_1 - Y_2}\tilde{\gamma} \in \mathcal{S}_2(\Omega_R, \gamma, Y)$. This shows that S_0 is independent of the choice of Y .

Finally, we show that S_0 is independent of the choice of cone point $\{0\}$. Two different choices lead to two cone operators κ_1 and κ_2 . We deduce from Lemma 7.1.2 that $(\kappa_1 - \kappa_2)(S - P_Y\tilde{\gamma}) = 0$, showing that area is well defined. This is the only place where we use the operator κ .

The differential 2-chain S_0 is therefore a solution to the general problem of Plateau, proving Theorem 1.0.1.

Remarks 7.3.4.

- A closed frame β is defined to a union of closed curves. Any frame $\cup_{i=1}^s \gamma_i$ supports an integral dipole curve $C = \text{supp}(\sum_{i=1}^s P_{X_i}\tilde{\gamma}_i)$ where γ_i is smoothly embedded in \mathbb{R}^3 , X_i is a vector field whose component orthogonal to γ_i is unit. The X_i can be chosen so that $\partial C = \sum_{i=1}^s \partial P_{X_i}\tilde{\gamma}_i = 0$ (see Figure 4). Any finite number of junctions are permitted, and we still obtain a cycle. We may therefore apply our methods to find a spanning surface of a prescribed closed frame with minimal area (see Figure 7).

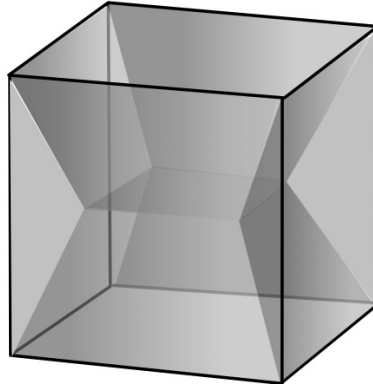


Figure 7: Plateau's square

- Smoothly embedded curves that are not cycles can have spanning surfaces as in Figure 8. The question arises⁵, what is the boundary of such a surface S ? Our methods apply to this surface,

⁵Almgren used this example in [Alm75] to defend the lack of a boundary operator for varifolds. He wrote, "in many of the phenomena to which our results are applicable there seems no natural notion of such an operator." —

and produce $\partial S = \partial E_X \tilde{\gamma}$ where γ is the part of the Jordan curve that meets $\text{supp}(S)$.

- *It is an interesting question to state and pose a version of Plateau's problem for frames which are defined as unions of smoothly embedded arcs which are not necessarily closed. The definition of "span" has to be reformulated as a first step. It would then be desirable to find a condition on an arc to guarantee existence of a nontrivial spanning surface.*

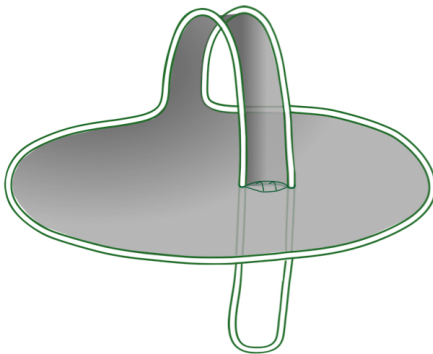


Figure 8: The boundary $\tilde{\gamma}$ of this film is a cycle, but its support γ is not.

- *Other constraints are possible using the continuous operators on chains available to us, not boundary. An intriguing example is $\perp \partial \perp$. These are topics for further research.*

References

- [Alm66] Frederick J. Almgren, *Plateau's problem, an invitation to varifold geometry*, Benjamin, 1966.
- [Alm75] ———, *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Bulletin of the American Mathematical Society **81** (1975), no. 1, 151–154.
- [Alt73] Hans Wilhelm Alt, *Verzweigungspunkte von h -flächen. ii.*, Math. Ann. **201** (1973), 33–55.
- [Dav10] Guy David, *$C^{-(1+\alpha)}$ regularity for two-dimensional almost-minimal sets in \mathbb{R}^n* , Journal of Geometric Analysis **20** (2010), 837–954.
- [Dou31] Jesse Douglas, *Solutions of the problem of Plateau*, Transactions of the American Mathematical Society **33** (1931), 263–321.
- [Fed69] Herbert Federer, *Geometric measure theory*, Springer, Berlin, 1969.
- [FF60] Herbert Federer and Wendell H. Fleming, *Normal and integral currents*, The Annals of Mathematics **72** (1960), no. 3, 458–520.
- [Fle62] Wendell H. Fleming, *On the oriented Plateau problem*, Rendiconti del Circolo Matematico di Palermo **11** (1962), no. 1, 69–90.
- [Fle66] ———, *Flat chains over a finite coefficient group*, Transactions of the American Mathematical Society **121** (1966), no. 1, 160–186.
- [Gul73] Robert Gulliver, *Regularity of minimizing surfaces of prescribed mean curvature*, Annals of Mathematics **97** (1973), no. 2, 275–305.
- [Har04a] Jenny Harrison, *Cartan's magic formula and soap film structures*, Journal of Geometric Analysis **14** (2004), no. 1, 47–61.
- [Har04b] ———, *On Plateau's problem for soap films with a bound on energy*, Journal of Geometric Analysis **14** (2004), no. 2, 319–329.
- [Har10] ———, *Operator calculus – the exterior differential complex*, submitted, December 2010.
- [HS79] Robert Hardt and Leon Simon, *Boundary regularity and embedded solutions for the oriented Plateau problem*, Bulletin of the American Mathematical Society **1** (1979), no. 1, 263–265.

- [Mor88] Frank Morgan, *Geometric measure theory: A beginners guide*, Academic Press, London, 1988.
- [Oss70] Robert Osserman, *A proof of the regularity everywhere of the classical solution to Plateau's problem*, *Annals of Mathematics* **91** (1970), 550–569.
- [Pla73] Joseph Plateau, *Experimental and theoretical statics of liquids subject to molecular forces only*, Gauthier-Villars, 1873.
- [Rei60] E.R. Reifenberg, *Solution of the plateau problem for m -dimensional surfaces of varying topological type*, *Acta Mathematica* **80** (1960), no. 2, 1–14.
- [Tay76] Jean Taylor, *The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, *Annals of Mathematics* **103** (1976), no. 2, 489–539.
- [Whi57] Hassler Whitney, *Geometric integration theory*, Princeton University Press, Princeton, NJ, 1957.
- [Zie62] W.P. Ziemer, *Integral currents mod 2*, *Transactions of the American Mathematical Society* **105** (1962), 496–524.
- [Zie69] William P. Ziemer, *Plateau's problem: an invitation to varifold geometry*, *Bulletin of the American Mathematical Society* (1969), 924–925.